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INFINITE AND IMAGINARY ELEMENTS IN ALGEBRA AND GEOMETRY.

By R. M. WINGER, University of Washington.

1. Introduction. Imaginary Points in Analytic Geometry. The traditional treatment of imaginary and infinite elements in algebra and geometry has curiously resisted the reform movement in American textbook writing. While imaginary numbers are receiving perhaps more attention in the algebra books, the imaginary point is still an outlaw.[1] The equation $x^2 + y^2 = 0$ "represents a single point or point circle"[2] while the equation $x^2 + y^2 + 1 = 0$ "represents no locus whatever." This is all the more remarkable in view of the historical fact that it was the geometric representation of imaginaries that gave these numbers their algebraic standing. Probably these same writers would not hesitate to speak of a point in space of four dimensions as a geometric image of a tetrad of numbers although such a point is as impossible to plot as one whose Cartesian coördinates in the plane are imaginary. Now that modern pure geometry has found a way to introduce imaginary points independent of algebra, it would seem absurd not to utilize the algebraic approach which is more elementary.

A reciprocal custom prevails with respect to infinity. Modern geometry has found a line at infinity indispensable as a bond of union between projective and metric properties, to render the principle of duality universally valid and to preserve a (1, 1) correspondence between a figure and its projection. Geometry has also found the number ∞ serviceable in the analytic theory of collineations, the parametric representation of curves and in studying the behavior of curves at infinity. But algebra steadfastly refuses to avail itself of these conveniences. True in the theory of equations in one variable the conditions for infinite roots are frequently given, but no mention is made of infinite roots of simultaneous equations where the idea is most valuable.

Authors of college algebras may hesitate to increase the formidable multiplicity of topics or they may regard the difficulties insuperable to freshmen. But these authors do not scruple to encroach on the fields of analytic geometry and even calculus to an extent that important algebraic material is crowded out altogether.[3] On the other hand, projective geometry has usually been presented synthetically so that any algebraic discussion would appear artificial. Careful writers on both subjects follow the sound practice—under the limitations they have prescribed—of telling the truth and nothing but the truth even if it is impossible to tell the whole truth. An occasional author however fails to tell the strict truth, as for example when he says that a linear and a quadratic equation in two variables *always* have two solutions or that two quadratics always have four. And I am not sure that the writer who says that the locus of the equation $x^2 + y^2 = 0$ is a single point does not convey the impression that he is telling a general truth rather than a truth restricted to the real domain. For he invariably uses "point" and "locus" to signify "real point" and "real locus" without troubling himself to explain his meaning.

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[1] Exception must be made of the excellent book by Snyder and Sisam, *Analytic Geometry of Space*. Even here after discussing imaginary elements the authors lapse into the old form of statement,—that a tangent plane cuts an ellipsoid in a "point ellipse," *e.g.*, rather than in a pair of imaginary lines. Crawley and Evans, *Analytic Geometry*, is the only freshman text that has come to my notice which recognizes imaginary points.

[2] This statement always recalls a bizarre work on *Algeometry* in which the author declares that while $x^2 + y^2 = 0$ is the equation of a *round* point no one has been able to write the equation of a *square* point!

[3] One recent book of 250 pages devotes nearly a third of its space to calculus, both differential and integral.

Certainly in analytic geometry where the streams of algebra and geometry merge there would seem to be the least excuse for these half-truths. For nowhere is the interplay between algebra and geometry more beautifully exemplified than in the theories of infinite and imaginary elements. The college student should come to the study of analytic geometry with a knowledge of imaginary numbers which, thanks to such books as Wilczynski's,[1] have been freed of their stigma and made genuinely real in his experience. Imaginary solutions of equations have been accepted on the same terms as real solutions. He has been taught that a quadratic equation in one variable always has two roots, real or imaginary, distinct or equal. Why then should he be told that certain equations have no loci and that equations with imaginary solutions represent curves which do not intersect, if indeed they represent curves at all? Why not take advantage of the ground already gained in algebra and say that a line and a circle in the plane always meet in two points, real and distinct, real and coincident or conjugate imaginary? The new statement not only serves every purpose of the old "two points, one point, no point" terminology but is actually more descriptive of the relation of the line and circle. A generator of a circular cone is a true example of a line that meets the circle of the base in but one point while the axis of the cone meets the circle in no point.

Absolute coördinates. It will be instructive to consider briefly a system of metrical coördinates in which the conventional rôles of real and imaginary are varied. We refer to the absolute coördinates used with such signal success by Professor Morley and his students in discussing metrical properties. In this system the bilinear axes are the circular rays through the Cartesian origin. If X and Y are the rectangular coördinates of a point, the absolute coördinates x, $\bar{x}$ are defined to be

$$x = X + iY, \qquad \bar{x} = X - iY, \qquad i = \sqrt{-1}. \tag{1}$$

The absolute coördinates, being complex, are represented geometrically in the usual manner on the complex plane. But if plotted separately x and $\bar{x}$ would in general determine two points. To avoid ambiguity we plot only the x, although the $\bar{x}$ cannot be ignored. When the Cartesian coördinates are real the absolute coördinates are in general imaginary. The converse is however not generally true. Neither does the reality of the absolute coördinates imply reality of the corresponding point. Thus on the line

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[1] *College Algebra with Applications*, Boston, 1916.

$$x + \bar{x} = 2, \qquad \text{or} \qquad X = 1, \tag{2}$$

those points are real whose x-coördinates are of the form $1 + iY$. The corresponding $\bar{x}$-coördinates are then $1 - iY$. A single real point of the line has real absolute coördinates, viz., the point (1, 1). But while the number pairs (2, 0), (3, − 1), $(1 + \sqrt{2}, 1 - \sqrt{2})$, $(i, 2 - i)$ satisfy the equation of the line, they cannot be represented. For if we plot the x alone the first three points would all lie on the line $x - \bar{x} = 0$ whereas this line cuts (2) only in the point (1, 1). If the algebra and geometry are to correspond, we must say that these pairs of numbers are coördinates of *imaginary* points on the line. In this system we have the following criterion:

A point is real or imaginary according as its absolute coördinates are or are not conjugate complex numbers.[1]

Again the parametric representation of curves in absolute coördinates differs from that in Cartesian coördinates. In the latter system real parameters are assigned to real points. In absolute coördinates, on the other hand, if the coefficients in the equations are real, the parameters of real points will be imaginary in general. Instead of thinking of the parameter as running along a line and ranging through all values, we may suppose the parameter to run around the unit circle in the complex plane, *i.e.*, the parameter t is a complex number of absolute value 1. The conjugate of t is then $1/t$. Thus the parametric equations of the unit circle are

$$x = t, \qquad \bar{x} = 1/t, \tag{3}$$

for the x-coördinate of a point is the same as the parameter. The only real parameters attached to real points of the circle are ± 1 which are cut out by the line $x - \bar{x} = 0$.

We need go no farther in this development here. Enough has been said perhaps to indicate the danger of dismissing imaginary coördinates as representing no points. The introduction of imaginary points into analytic geometry obviates a multitude of exceptions which are distasteful to a geometer who believes that his subject is every whit as good as algebra.[2]

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[1] Complex refers to either real or imaginary numbers, a real number being self-conjugate.

[2] One has but to glance through the recent book of Osgood and Graustein, *Plane and Solid Analytic Geometry,* to see how many such exceptions must be noted by an exacting writer. A conspicuous example is the important theorem (pp. 167–8) relating to a pencil of curves $u + kv = 0$. They divide the theorem into two parts according as $u = 0$ and $v = 0$ do or do not intersect. The first part has an exception when $u + kv = 0$ represents a single point (or a finite number of points). The second part reads: "Let $u = 0$ and $v = 0$ be the equations of two non-intersecting curves. Then the equation $u + kv = 0$ $(k \neq 0)$ represents, in general, a curve not meeting either of the two curves." The beautiful positive theorem (part one) is thus replaced by a weak and purely negative theorem devoid of its chief interest. Moreover two exceptions are given (in addition to $k \neq 0$), viz., $u + kv = 0$ may represent (*a*) a single point, (*b*) no locus. Even then (*b*) must stand for such diverse cases as constant = 0 and $x^2 + y^2 + 2 = 0$. Finally a special case is given when $u = 0$ and $v = 0$ represent parallel lines. How much more satisfactory, having annexed the infinite and imaginary domain, to be able to say: $u + kv = 0$ (*always*) represents a curve through all the common points of $u = 0$ and $v = 0$, and, if $k \neq 0$ or ∞, through no other

2. The Number Infinity in Algebra. On the other hand, the algebraist who believes his subject is as good as geometry will be distressed at the failure of his fraternity to appropriate the advantage won by the geometer in the introduction of infinite elements. The geometer has found it convenient to postulate a point at infinity on every line or a line at infinity in every plane.[1] The most immediate consequence is that parallel lines may then be defined as lines which meet at infinity.[2] How can we make use of this definition and postulate in algebra? Perhaps the best approach is through homogeneous equations when infinite roots are placed on the same basis as zero roots. But we need not resort to homogeneous equations. We might begin by postulating a number ∞—call it an improper number if you like—which shall be the coördinate, in a Cartesian coördinate system along a line, of the "improper" point at infinity on the line.[3] This number is now clothed with properties to conform to those of its geometric counterpart. Thus the x-intercept of the line

$$y = mx - k \tag{4}$$

is

$$x_0 = k/m. \tag{5}$$

Now if $m = 0$, the line is parallel to the x-axis and cuts it therefore at infinity. Accordingly in virtue of (5) we attribute to ∞ the property

$$k/0 = \infty, \qquad k \neq 0, \tag{6}$$

for if $k = 0$ at the same time, the line (1) coincides with the x-axis and intersects it at every point.[4] It is then but a step to the theorem that the equation

$$a_0x^n + a_1x^{n-1} + \cdots + a_rx^{n-r} + \cdots + a_{n-1}x + a_n = 0 \tag{7}$$

has r infinite roots if $a_0 = a_1 = \cdots = a_{r-1} = 0$, so that an equation of apparent degree r may be regarded as an equation of degree n with $n - r$ roots infinite.

Although some writers explain that repeated and imaginary roots must be counted if an equation of degree n is to have n roots, few authors include infinite roots in the enumeration on the ground that the degree of the equation has been

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point of either curve. The trivial exception here applies only to the parameter whereas the exceptions in the text restrict the base curves themselves. Again one objects to a definition (p. 312) which confines the polar of a point outside a conic to the line segment cut off by the conic —especially when its equation is given, for the equation certainly represents the whole line. These and kindred exceptions due to the refusal of the authors to countenance imaginary and infinite elements and to assign values to indeterminates mar, in my judgement, an otherwise admirable book.

[1] I wish to insist that this is not projective geometry. Any geometry that isolates a line at infinity and invests it with exceptional properties is not projective. The line at infinity is considered in projective geometry because it serves as a bridge between projective and Euclidean metric properties.

[2] This definition suffices for either the plane or space since two intersecting lines necessarily lie in the same plane.

[3] This is precisely what is done in Veblen and Young, *Projective Geometry*, vol. 1, chapter 7—EDITOR.

[4] Other properties may be assigned by considering the intercepts of the parabola $y = ax^2 + bx + c$, but this will suffice for our present purpose.

reduced. The recognition of infinite roots is however no less essential. For example, the student has learned to expect two solutions of a linear and a quadratic equation even though the roots be coincident or imaginary. Then he encounters two equations like

$$x^2 - 9y^2 = 7, \quad (8) \qquad\qquad x = 3y + 1, \quad (9)$$

which on combination lead to

$$9y^2 - 9y^2 + 6y = 6, \tag{10}$$

whence $y = 1$, $x = 4$. Again if he attempts to solve (8) with

$$x = 3y, \tag{11}$$

he will get

$$9y^2 - 9y^2 = 0 = 7, \tag{12}$$

a traditional absurdity. Do (8) and (9) then have a single solution while (8) and (11) have none? This will be the view of his textbook. The orthodox algebraist must say—if he does not avoid the cases in question—that pairs of linear and quadratic equations in two variables and with real coefficients fall into five classes as follows: those with two solutions which are (*a*) real and distinct, (*b*) real and equal, (*c*) conjugate imaginary, (*d*) those with one real solution, and (*e*) those with no solution whatever,[1] while his hyper-orthodox geometrical brother would say that a line cuts a conic in two points, one point or no point. In a sense the algebraic statement is above criticism—it is largely a question of the point of view. But the geometrical statement is not adequate since it affords no criterion for distinguishing a tangent from a line parallel to an asymptote, nor an asymptote from other lines which "do not cut the conic." Furthermore, it places tangent and asymptote in entirely different categories whereas an asymptote might at least be defined as the limit of a tangent. The heterodox but progressive analytic geometer would say that *a linear equation and a quadratic always have two solutions and that a line always cuts a conic in two points*—real or imaginary, distinct or coincident, finite or infinite—thus placing algebra and geometry on an equal footing. Only by recognizing infinite as well as imaginary roots and both imaginary and infinite intersections can this theorem be generalized to two equations (or curves) of degrees m and n.

But we need not go to simultaneous equations and the geometry of two dimensions nor to the infinite region of the plane to find a sound reason for introducing infinite elements. The recognition of infinite elements in the geometry on a curve, which is essentially a geometry of one dimension, is extremely useful. For example, the equation of a circle with radius 1 and center at the origin can be written parametrically in rectangular coördinates

$$\begin{aligned} x &= \cos\theta = (1 - t^2)/(1 + t^2), \\ y &= \sin\theta = 2t/(1 + t^2), \end{aligned} \tag{13}$$

where $t = \tan(\theta/2)$. To find the intersections with the x-axis, we set $y = 0$

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[1] If imaginary coefficients are taken into account, he would have even more varieties.

whence $t = 0$, $x = 1$. But obviously the x-axis cuts the circle in a second point $(-1, 0)$. We must say then (a) there is one point on the circle with no parameter, (b) equations (13) represent a circle exclusive of the point $(-1, 0)$ or (c) the x-axis cuts the circle in a second point with parameter $t = \infty$. Is there anyone who will not admit that on considerations of continuity, generality and elegance the last statement is superior? Isn't it more in harmony with the spirit of modern mathematics and is it any less intelligible to the student?

It frequently happens in Cartesian geometry when curves are written parametrically that the parameter ∞ is attached to a point in the finite part of the plane. Another familiar instance is the folium of Descartes,

$$x = \frac{at}{t^3 + 1}, \qquad y = \frac{at^2}{t^3 + 1},$$

the double point of which is located at the origin but has the parameters 0 and ∞. If θ is the parameter on a curve and x and y are rational functions of $\sin\theta$, $\cos\theta$, $\tan\theta$, then the coördinates of the points on the curve can be expressed as rational functions of $\tan\theta/2$ by means of the relations in (13). There will be one point on the curve with the parameter ∞ and this point is as likely to be in the finite as in the infinite region.

3. Infinite Roots and Inconsistent Equations. The solution of the simultaneous equations:

$$\begin{aligned} a_1x + b_1y + c_1 &= 0, \\ a_2x + b_2y + c_2 &= 0, \end{aligned} \tag{14}$$

may be written

$$x = \frac{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \equiv \frac{A}{C}, \qquad y = \frac{\begin{vmatrix} c_1 & a_1 \\ c_2 & a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \equiv \frac{B}{C}. \tag{15}$$

Obviously the solution is unique unless $C = 0$. If we examine the possibilities when $C = 0$, we recognize two cases: (I) A and B not both $= 0$ and (II) $A = B = 0$.

Case I. $C = 0$, $A \neq 0$. The equations are commonly said to have no solution. There are three characteristic hypotheses depending on the way in which C becomes zero.

(a) $a_1 = b_1 = 0$. Then $A = -b_2c_1$, $B = c_1a_2$. The intercepts of the first line are $-c_1/a_1$ and $-c_1/b_1$ which under our hypothesis become infinite and the equation of the first line takes the form

$$0x + 0y + 1 = 0. \tag{16}$$

In other words the line has moved off to infinity and (16) may be regarded as the equation of the line at infinity. Since A is different from zero $b_2, c_1 \neq 0$ and the other line remains finite. The two lines thus meet at infinity,—at the infinitely distant point on the second line.

(*b*) $a_1 = a_2 = 0$. Then $A = b_1c_2 - b_2c_1$, $B = 0$ and the two lines reduce to $b_1y + c_1 = 0$ and $b_2y + c_2 = 0$. Since $A \neq 0$ the two lines exist and remain distinct. They are manifestly parallel to the x-axis (or one is parallel to the x-axis and the other coincident with it) and they meet therefore at the infinitely distant point of that axis.

(*c*) $a_1/a_2 = b_1/b_2 \neq c_1/c_2$. Barring hypotheses (*a*) and (*b*) the equations assume the form

$$a_1x + b_1y + c_1 = 0, \qquad a_1x + b_1y + c_2 = 0,$$

whose solution is $x = \infty$, $y = \infty$. The lines are parallel and meet at infinity.

Under each hypothesis of case I we have found that the two equations may be interpreted as representing distinct lines which meet in a point at infinity. The coördinates of this point will be a solution of the equations and we no longer have pairs of inconsistent linear equations. Or we may retain the term "inconsistent," introducing the definition: The two linear equations (14) are inconsistent when and only when they represent lines which meet at infinity.

Under case II, $A = B = C = 0$ when the lines coincide and the equations have a single infinity of solutions. If however $a_1 = a_2 = b_1 = b_2 = 0$, the lines coincide with the line at infinity and the equations would then be "inconsistent."

Three linear equations. Similarly the solutions of the three equations

$$a_ix + b_iy + c_iz + d_i = 0, \qquad i = 1, 2, 3, \tag{17}$$

may be written $x = A/D$, $y = B/D$, $z = C/D$. If $D = 0$ but not all the numerators are zero, the values of one, two, or all of the unknowns will be infinite. The equations will then be "inconsistent" as before. Geometrically, the three planes meet in a unique point on the plane at infinity.[1]

If $A = B = C = D = 0$, the equations are dependent and have ∞^1 or ∞^2 solutions according as the planes have a common line or coincide. Such equations however may have "no solution" according to the orthodox view, as, *e.g.*, the three equations

$$ax + by + cz = d_i, \qquad i = 1, 2, 3, \qquad d_1 \neq d_2 \neq d_3. \tag{18}$$

These three planes are parallel and hence have their infinitely distant lines in common. We are led to extend our definition of inconsistent equations as follows: Three non-homogeneous linear equations in three variables, whether dependent or independent, are inconsistent when and only when they represent planes which meet only at infinity.

A general definition of inconsistent equations. These results may be generalized for a set of n linear equations in n variables.[2] Indeed the definition is valid when there are fewer equations than variables. Thus the equations of two parallel planes would be inconsistent. Furthermore any set of equations in linear form will be inconsistent under similar conditions. For example, the equations

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[1] There are three possibilities: (1) one plane cuts the other two and is parallel to their line of intersection, (2) two of the planes are parallel and the third a finite plane, (3) two planes meet in a finite line while the third is the plane at infinity.

[2] This demands of course the assumption that the locus of points at infinity in a linear space S_n of n dimensions is an S_{n-1}.

$$\begin{aligned} ax^2 + by^2 + c_1 &= 0, \\ ax^2 + by^2 + c_2 &= 0, \end{aligned} \qquad c_1 \neq c_2, \tag{19}$$

are linear in x^2 and y^2. Subtracting we should get constant $= 0$ so that the equations are inconsistent in the usual sense. But using formula (15) we find $x^2 = \infty$, $y^2 = \infty$ and we may say that the equations have two pairs of equal roots, *i.e.*, the curves have double contact at infinity.

Two equations need not be of the same degree to be inconsistent. An instance of this already noticed is the equation of a hyperbola and that of its asymptote. Another example is furnished by the equations

$$\begin{aligned} y^3 - xy^2 - x^2 + 2xy &= 0, \\ y^2 - y + x + 1 &= 0, \end{aligned} \tag{20}$$

which, on elimination of x, yield

$$0y^4 + 0y^3 + 0y^2 + 0y - 1 = 0. \tag{21}$$

The equations which should have six solutions thus have no finite solutions and the corresponding curves meet wholly at infinity. We may now formulate a general criterion for inconsistent equations:

r non-homogeneous (dependent or independent) equations in n variables, $r \leqq n$, are inconsistent when and only when the loci of the equations intersect, whether in real or imaginary points, wholly at infinity.

The purpose of this discussion has been not merely to criticize the shortcomings of our current elementary textbooks in respect to imaginary and infinite elements but to illustrate how these fruitful ideas can be employed to enrich both algebra and geometry. The suggestions here embodied are consonant with the spirit that pervades modern mathematics and with sound European tradition. It is my conviction that they could be incorporated into our elementary texts without sacrificing either clearness or rigor and at no greater cost of space than is required to detail the exceptions which they eliminate.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

26. BURCKHARDT ON MODERN TEACHING.

The number of times the teaching of mathematics has been reformed and the general similarity of view of the reformers are always interesting to the student of the history of the subject. Every day, in educational circles, theories are launched forth that have been common property so long that it is not to be wondered that, in certain schools, the history of education is frowned upon,